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The excess noise field of subsonic jets

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The sound field generated by the interaction of spatial instabilities on the shear layer shed from a duct with the nozzle lip is studied. It is shown that the intensity varies with direction θ from the exhaust and with the subsonic exhaust speed U according to $I \sim U^6(1-\cos\theta)^2$ and $I \sim U^6 \sin^2\theta$ for the axisymmetric and first azimuthal (sinuous) modes respectively. The first of these results is interpreted in terms of monopole and dipole sources at the exit plane, representing the acoustic effect of fluctuating mass flow and axial thrust across the exit plane, and the second in terms of a transverse dipole at the exit plane, corresponding to fluctuations in cross-stream thrust. A correlated thrust fluctuation of 1 % is shown to overwhelm the jet mixing noise in the forward arc, $\theta > 90°$, while the acoustic efficiency of the interaction process is never less than $10^{-6}M^3$ even under the cleanest possible exit conditions. Forward flight of the duct at Mach number M_a is shown to increase the forward-arc intensity by the factor $(1+M_a\cos\theta)^{-4}$. It is suggested that much of the discrepancy between the noise fields of real engines and the predictions of Lighthill's theory of jet mixing noise – the so-called 'excess noise' problem – can be explained in terms of this interaction mechanism.

1. Introduction

Current trends towards very high by-pass ratios in jet transport turbofan engines have focused attention again on the problem of jet noise at low exhaust speeds. This should be an area adequately covered by the Lighthill (1952, 1954) theory of jet mixing noise, since that theory is an asymptotic one whose validity improves as the Mach number decreases. Indeed, in 1963 Lighthill was able to give a convincing demonstration of the relevance of his theory to subsonic jet noise. More recent experiments have, however, revealed quite substantial deviations under certain circumstances from the predictions of Lighthill's theory, small-scale experiments and full-scale tests alike showing rather similar deviations in model rigs, in high-speed turbojet engines operating at reduced power levels and in modern turbofan engines. This situation has become known in England as the 'excess noise' problem. For background on the way in which it has arisen, and for the theory of a number of possible contributory mechanisms, the reader is referred to Ffowcs Williams *et al.* (1972). The present paper reports a continuation of that work.

No doubt there are many physical processes capable of producing 'excess noise' in readily measurable quantities under appropriate conditions. It is an intriguing aspect of excess noise, however, that it appears to be dominated by a particular sound field with more or less universal features. Thus, for example, shock–turbulence interaction cannot be responsible for the whole excess noise field of an imperfectly expanded supersonic jet, for that excess noise field is very similar to that of a properly expanded jet or, for that matter, to that of a subsonic jet. We shall use the term 'excess noise' here to signify a field with the following general characteristics.

(*a*) It involves an index for the variation of intensity with exhaust speed U between 4 and 6, to be contrasted with the eighth-power variation of mixing noise (Lighthill 1952). (Indices as low as 2 have also been reported to us, but always under rather extreme conditions not likely to be met in practice.)

(*b*) It has a pronounced forward directivity. In contrast, mixing noise peaks in the rear arc, at around 45° to the exhaust for both subsonic and supersonic jets (Lighthill 1963).

(*c*) It peaks at a frequency rather higher than that characteristic of mixing noise. A factor of between, say, 4 and 10 is involved here, the spectra being generally too flat for a more decisive statement.

(*d*) In the forward arc, the total acoustic intensity in the frequency band associated with excess noise is comparable with that in the frequency band associated with mixing noise.

(*e*) The forward-arc excess noise is *absolutely increased* by forward flight of the aircraft, the increase in intensity being around 3 dB at 150° to the exhaust and 6 dB dead ahead in a typical landing approach at a flight Mach number of 0·3. In contrast the rear-arc mixing noise is decreased by the seventh power of the ratio of jet exhaust speed relative to the surrounding fluid to the jet exhaust speed relative to the nozzle.

It is easy to think of mechanisms producing a field with some of the attributes (*a*)–(*e*); we have found only one with all of those properties.

As an example, shock-cell noise in an imperfectly expanded supersonic jet is often claimed to have a pronounced forward directivity (see, e.g. Lighthill 1963), though there are several possible points of contention in the argument. It may well also have features (*a*), (*c*) and (*e*), though at present there is no theoretical evidence to support such a claim. Moreover, the shock-cell/turbulence interaction constitutes a sound source fixed relative to the jet nozzle, and so subject to Doppler amplification in the forward arc under forward-flight conditions. However, the shock strengths are reduced by forward flight; the change is small under conditions typical of aircraft landings or take-offs, but is enough to almost completely offset the Doppler increase, and so to rule out feature (*e*) above. As another example, fluctuating thrust levels of the order of 1–2 % can be shown to produce a sound field with features (*a*), (*c*), (*d*) and (*e*). But, apparently, fluctuating thrust would constitute an acoustic dipole at the exit plane, with a sound field peaking in the downstream direction, and thus not exhibiting feature (*b*). We shall see, however, that this conclusion as to the directivity associated with fluctuating thrust is in error, and that this mechanism – or more generally, unsteady flow interaction with the jet tailpipe – does indeed have all the required features.

It is almost essential to come to this conclusion in a rather indirect way, motivated by an extension of previous work (Crighton 1972). There the author followed the work of Orszag & Crow (1970) on the instability of a vortex sheet leaving a large plate. Orszag & Crow discussed the modification caused by the inhomogeneous surface to the spatial Helmholtz eigenfunctions of the vortex sheet in incompressible flow, while the author derived expressions for the sound field resulting from this modification, both with and without the application of a Kutta condition to the unsteady trailing-edge flow. In particular, it was shown that the intensity–directivity law for the case of a vortex sheet leaving a rigid plate and developing a two-dimensional instability (under no application of a Kutta condition) was $I \sim U^4 \sin^2 \frac{1}{2}\theta$, U being the flow velocity on one side of the plate and θ being measured from the extension of the plate. For a three-dimensional disturbance we replace U^4 by U^5. This sound field is essentially independent of the highly nonlinear flow which eventually develops as the instability grows with downstream distance, and results from the influence of the plate on a small region of the downstream flow. It obviously exhibits aspects (*a*) and (*b*) of the excess noise field; arrangements were also presented in favour of aspects (*c*) and (*d*). At any rate, interaction of shear-layer instability with a large plane surface seemed to have sufficiently promising characteristics to prompt the extension of the model to the case of the interaction of instabilities on the cylindrical shear layer shed from a circular duct with the duct lip.

We pursue that objective here with a minimum of mathematical detail. The steps are standard in the Wiener–Hopf technique (Noble 1958), and are discussed at length, and in a simpler context, by Orszag & Crow (1970) and Crighton (1972).

2. Axisymmetric instability modes

Uniform flow at speed U issues from a semi-infinite hard-walled circular duct lying in $r = b$, $-\infty < x < 0$, forming a cylindrical vortex sheet in $r = b$, $0 < x < \infty$ between the jet flow and the stagnant fluid in $r > b$. We assume that there exists a steady-state linearized perturbation field to this basic flow, and a time factor $\exp(-i\omega t)$, $\omega > 0$, will be suppressed throughout. Denote the perturbation potentials in $r > b$ and $r < b$ by $\phi^{(1)}(x, r, \chi)$ and $\phi^{(2)}(x, r, \chi)$ respectively, χ being the azimuthal co-ordinate. These satisfy the Helmholtz equations

$$\left.\begin{aligned}(\nabla^2 + k_0^2)\,\phi^{(1)} &= 0,\\ [\nabla^2 - (M\,\partial/\partial x - ik_0)^2]\,\phi^{(2)} &= 0,\end{aligned}\right\} \tag{2.1}$$

with the boundary conditions on the duct

$$\partial\phi^{(1)}/\partial r = \partial\phi^{(2)}/\partial r = 0 \quad (r = b,\ -\infty < x < 0). \tag{2.2}$$

On the vortex sheet, the requirements of continuity of particle displacement and of pressure supply the conditions

$$\left.\begin{aligned}-i\omega\eta &= \partial\phi^{(1)}/\partial r,\\ (-i\omega + U\,\partial/\partial x)\,\eta &= \partial\phi^{(2)}/\partial r,\\ (-i\omega + U\,\partial/\partial x)\,\phi^{(2)} &= -i\omega\phi^{(1)},\end{aligned}\right\} \quad r = b,\ 0 < x < \infty, \tag{2.3}$$

$\eta(x, \chi)$ denoting the displacement of the vortex sheet from its mean position $r = b$. We have taken the mean density and sound speed to have the same values, ρ_0 and a_0, everywhere, and written M for the Mach number U/a_0, assumed less than unity. The acoustic wavenumber $k_0 = \omega/a_0$ is taken to have a small positive imaginary part, in accordance with a useful convention (Noble 1958, p. 29).

We follow the procedure of Orszag & Crow (1970), isolating contributions to the potentials which represent a spatial Helmholtz instability on an infinite cylindrical vortex sheet, and using the Wiener–Hopf method to determine the correction field demanded by the presence of the duct. The spatial Helmholtz eigenfunctions for an *axisymmetric* disturbance (to which we now restrict ourselves) are

$$\left.\begin{aligned} \phi^{(1)} &= A\exp(-i\alpha x)\,K_0(\gamma_\alpha r),\\ \phi^{(2)} &= B\exp(-i\alpha x)\,I_0(\varpi_\alpha r),\\ \eta &= d\exp(-i\alpha x), \end{aligned}\right\} \tag{2.4}$$

where the ratios $A:B:d$ and the eigenvalue equation

$$\gamma_\alpha D_\alpha^2 I_0(\varpi_\alpha b)\,K_0'(\gamma_\alpha b) - \varpi_\alpha I_0'(\varpi_\alpha b) K_0(\gamma_\alpha b) = 0 \tag{2.5}$$

are found by applying conditions (2.3) on $r = b$ for *all* x. Branch cuts are chosen so that $\operatorname{Re}\gamma_\alpha \equiv \operatorname{Re}(\alpha^2 - k_0^2)^{\frac{1}{2}}$ is positive for all complex α, while any branch may be chosen for $\varpi_\alpha \equiv \{\alpha^2 - (k_0 + M\alpha)^2\}^{\frac{1}{2}}$. D_α denotes $(1 + M\alpha/k_0)$, and (2.5) is a simple generalization of the eigenvalue equation first found by Batchelor & Gill (1962) in the context of temporal instability of incompressible jet flow. For $M < 1$ and any value of the Strouhal number $S = \omega b/U$, equation (2.5) has a root with $\operatorname{Re}\alpha < 0$, $\operatorname{Im}\alpha > 0$, representing an instability growing as it propagates down stream, and it is to that root that the symbol α will subsequently refer.

Now we add to the fields (2.4) correction fields ϕ, ψ and ζ respectively, these being such as to ensure satisfaction by the total fields of the mixed boundary conditions (2.2) and (2.3). The formal steps by which this is accomplished are described elsewhere (Orszag & Crow 1970; Crighton 1972). Defining

$$\Phi_+(s,r) = \int_0^\infty \phi(x,r)\,e^{isx}\,dx, \qquad \Phi_-(s,r) = \int_{-\infty}^0 \phi(x,r)\,e^{isx}\,dx \tag{2.6}$$

with corresponding expressions for the transforms $\Psi(s,r)$ of $\psi(x,r)$ and $Z(s)$ of $\zeta(x)$, we can derive a Wiener–Hopf equation

$$-i\omega Z_+(s) = K(s)\,F_-(s) - \omega d/(s-\alpha), \tag{2.7}$$

in which
$$K(s) = \frac{\gamma_s \varpi_s K_0'(\gamma_s b)\,I_0'(\varpi_s b)}{\gamma_s D_s^2 I_0(\varpi_s b)\,K_0'(\gamma_s b) - \varpi_s I_0'(\varpi_s b)\,K_0(\gamma_s b)} \tag{2.8}$$

and
$$F_-(s) = D_s \Psi_-(s,b) - \Phi_-(s,b) + (iU/\omega)\,\psi(0,b).$$

The solution of (2.7) is

$$-i\omega Z_+(s) = -\frac{\omega d}{s-\alpha}\left(1 - \frac{K_+(s)}{K_+(\alpha)}\right), \tag{2.9}$$

in terms of a factorization $K(s) = K_+(s)\,K_-(s)$, in which the factors are analytic and non-zero in overlapping upper and lower half-planes $\operatorname{Im}s > -(\operatorname{Im}k_0)/(1+M)$

and $\operatorname{Im} s < \operatorname{Im} k_0$ respectively ($\operatorname{Im} k_0$ being chosen to have a suitably small value, depending upon the value of S) and have the behaviour $K_{\pm}(s) = O(s^{-\frac{1}{2}})$ at infinity.

The field outside $r = b$ is given by

$$\phi = \frac{\omega d}{2\pi} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{K_+(s)}{K_+(\alpha)} \left(\frac{K_0(\gamma_s r)}{\gamma_s K_0'(\gamma_s b)} \right) \frac{e^{-isx}\, ds}{s-\alpha}, \tag{2.10}$$

with $-(\operatorname{Im} k_0)/(1+M) < \epsilon < +\operatorname{Im} k_0$.

Introducing polar co-ordinates (R, θ) such that $x = R\cos\theta, r = R\sin\theta, 0 \leqslant \theta \leqslant \pi$, the far field may be obtained from (2.10) by a steepest-descent calculation for $k_0 R \to \infty$, which leads at once to

$$\phi \sim -\frac{i\omega d}{2} \left(\frac{e^{ik_0 R}}{R} \right) \left(\frac{K_+(-k_0 \cos\theta)}{K_+(\alpha)} \frac{1}{(\alpha + k_0 \cos\theta)} \right) \left(\frac{1}{k_0 \sin\theta K_0'(-ik_0 b \sin\theta)} \right), \tag{2.11}$$

there being no difficulties associated with poles near the steepest-descent path.

We emphasize that this field is due entirely to the interaction of instabilities on the vortex sheet with the duct, and does not include the primary field of the instabilities as given in (2.4). That field is poorly represented by a linear model with exponential spatial growth; its proper description is provided by Lighthill's theory (1952). On the other hand, the influence of the duct on the instability eigenmodes of the vortex sheet is very weak in a hydrodynamic sense. In the case of a plane vortex sheet leaving a plate, the influence of the plate vanishes at distances greater than about U/ω from the edge (Orszag & Crow 1970), and "the interaction between a jet column and nozzle is probably even weaker" (Crow & Champagne 1971, p. 567). Linear theory is probably adequate to deal with this small region, throughout which there is no amplification by as much as a factor e, and consequently we believe (2.11) to be the correct interaction field regardless of the nonlinear breakup of the jet column which eventually ensues and invalidates (2.4) at some greater distance from the exit plane.

Consider now the long acoustic wavelength limit $k_0 b \ll 1$, or equivalently, the low Mach number limit for any fixed Strouhal number S. Write $\tau = s/k_0$ in (2.8), and let $M \to 0$ holding τ and S fixed and $O(1)$. Then we find

$$K(s) \sim \tfrac{1}{2} k_0 MS (\tau^2 - 1). \tag{2.12}$$

This of course is merely a degenerate form of the classic Levine–Schwinger kernel for the circular duct diffraction problem with no mean flow (see, e.g. Noble 1958, p. 110 *et seq.*). Without restriction on S, our $K(s)$ must obviously tend to the Levine–Schwinger kernel provided that $s = O(k_0)$ as $M \to 0$, for on the acoustic scale the mean flow contributes only a negligible refraction effect. If in addition we have $k_0 b \ll 1$, i.e. $MS = o(1)$ as $M \to 0$, the Levine–Schwinger kernel simplifies further to the form (2.12). Factorization is now immediate:

$$K(s) = A_+(\tau) A_-(\tau),$$

where

$$A_{\pm}(\tau) = a_{\pm} (\tfrac{1}{2} k_0 MS)^{\frac{1}{2}} (\tau \pm 1 \pm i0) \tag{2.13}$$

and $a_{\pm}$ are constants (viewed on the scale of the acoustic wavenumber) such that $a_+ a_- = 1$. It can be shown that the split functions (2.13) can be matched to appropriate 'inner' split functions which describe an incompressible flow with two length scales b and U/ω, each of which is small compared with the acoustic wavelength.

It now follows from (2.13) that

$$K_+(-k_0\cos\theta) = \tfrac{1}{2}(1-\cos\theta)\,K_+(k_0),$$

and use of this expression in (2.11) then results in an expression for the radiated density field:

$$\rho' = \left(\frac{i\rho_0}{4}\right)\left(\frac{k_0^2 d}{\alpha}\right)\left(\frac{b}{R}\right)\left(\frac{K_+(k_0)}{K_+(\alpha)}\right)(1-\cos\theta)\,e^{ik_0R}. \tag{2.14}$$

The directivity $1-\cos\theta$ of the far-field density suggests that in this limit an interpretation in terms of monopole and dipole sources is possible. Of course, Kirchhoff's theorem always allows such an interpretation, but in general that interpretation is valueless when those sources are distributed over a large but inhomogeneous surface. In the present problem we can regard the far field as generated by pressure dipoles over the surface of the duct and by monopoles and dipoles over the exit plane. At high frequencies, estimation of the far field due to a rapidly varying distribution of pressure dipoles over the curved surface of the duct presents a serious problem. In the low frequency limit, however, we can argue that the pressures are essentially in phase all round the duct at any fixed axial station, and hence that the total integrated dipole strength over the curved surface vanishes as the frequency drops to zero. The dominant equivalent sources should, therefore, be associated solely with mass flux and force across the exit plane provided that $k_0 b \ll 1$.

Now for a completely general (periodic) azimuthal variation, only the axisymmetric mode carries a non-zero axial mass flux and axial force across any section through the duct. These quantities may therefore be quite generally evaluated from the solution (2.9), which yields the expression

$$\psi(x,r) = \frac{\omega d}{2\pi}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{D_s}{s-\alpha}\,\frac{K_+(s)}{K_+(\alpha)}\,\frac{I_0(\varpi_s r)}{\varpi_s I_0'(\varpi_s b)}\,e^{-isx}\,ds \tag{2.15}$$

for the correction field in $r < b$. For $x < 0$, we deform the path to infinity in the upper half-plane. There are no branch cuts, only poles at $s = \alpha$ and from the roots of $\varpi_s I_0'(\varpi_s b) = 0$. The pole at $s = \alpha$ gives a contribution

$$\psi_\alpha = -Be^{-i\alpha x} I_0(\varpi_\alpha r),$$

which cancels the primary field (2.4) of the unstable mode inside the duct. When $k_0 b \ll 1$, there is one pole at $s = k_0/(1-M)$, which generates a propagating mode in the duct, giving, in the low Mach number limit,

$$\psi = -\left(\frac{i\omega d}{\alpha}\right)\left(\frac{1}{k_0 b}\right)\left(\frac{K_+(k_0)}{K_+(\alpha)}\right)e^{-ik_0 x}. \tag{2.16}$$

All other poles give non-propagating modes which decay at least as rapidly as $\exp(-|x|/b)$ as $x \to -\infty$ in the duct.

We assert now that the mass flux and axial force across the exit plane are, to leading order, equal to those quantities evaluated across a distant upstream station, at which (2.16) applies. One can prove this statement by direct evaluation of the difference between the relevant quantities on the basis of (2.15), a method which, however, requires more knowledge of the split functions $K_{\pm}(s)$ than we otherwise need here. Alternatively, Kirchoff's theorem applied to a large spherical control surface cutting the duct at a distant upstream section allows us to express the far field in terms of the mass and momentum flux there, and since we have already argued that the far field is determined solely by those quantities at the exit plane it follows that the mass and momentum fluxes must be constant, to leading order, at all sections through the duct.

The mass flux Q and axial force F across the exit plane can therefore be found from (2.16) in the forms

$$Q = \rho_0 \int_0^b \frac{\partial \psi}{\partial x}(0,r)\, 2\pi r\, dr,$$

$$F = \int_0^b \left\{ p(0,r) + \rho_0 U \frac{\partial \psi}{\partial x}(0,r) \right\} 2\pi r\, dr$$

$$= -\rho_0 \int_0^b \frac{\partial \psi}{\partial t}(0,r)\, 2\pi r\, dr,$$

the pressure perturbation being $p = -\rho_0\,(\partial/\partial t + U\,\partial/\partial x)\,\psi$. The density fields to which these monopole and dipole sources would give rise are

$$\rho'_Q = \frac{1}{4\pi a_0^2 R} \frac{\partial Q}{\partial t}(t - R/a_0)$$

and

$$\rho'_F = \frac{1}{4\pi a_0^3 R} \frac{\partial F}{\partial t}(t - R/a_0)\cos\theta,$$

and it is easily seen that the sum $\rho'_Q + \rho'_F$ is exactly equal to the total field ρ' as given in (2.14).

We have therefore proved that (2.14) is equivalent to the statement that $\rho' = \rho'_Q + \rho'_F$, where Q and F are the mass and momentum fluxes across the exit plane, associated with the propagation of a plane wave $\exp(-ik_0 x - i\omega t)$ *up* the duct. A more appropriate form is

$$\rho' = \frac{-1}{4\pi a_0^3 R} \frac{\partial F}{\partial t}(t - R/a_0)(1 - \cos\theta), \tag{2.17}$$

which makes it clear that the monopole is $O(M)$ rather than $O(1)$, and that, because Q and F are involved in a wave propagating upstream, the weak monopole *cancels* the dipole downstream and reinforces it in the forward arc. In contrast, if Q and F were associated with a wave incident from upstream, we would then have a directivity $1 + \cos\theta$, with a maximum downstream, and little effect in the forward arc.

Suppose that the r.ms. value of F is $\epsilon\rho_0 U^2 A$, where A is the nozzle area, and denote by S the Strouhal number of disturbances which make the maximum contribution to $\overline{(\partial F/\partial t)^2}$. Then the intensity from flow–surface interaction is, according to (2.17),

$$I = \frac{A^2}{16\pi^2 b^2 R^2} \epsilon^2 S^2 (1 - \cos\theta)^2 \rho_0 U^3 M^3, \tag{2.18}$$

and the acoustic efficiency η = (total sound power)/(jet power $\frac{1}{2}\rho_0 U^3 A$) is given by

$$\eta = \tfrac{2}{3}\epsilon^2 S^2 M^3. \tag{2.19}$$

With $\epsilon = 1\,\%$ (i.e. a net r.m.s. fluctuating thrust level of 1 %) and S close to unity, as is indicated by the experiments of Crow & Champagne (1971), we have $\eta \sim 10^{-4} M^3$, which should be compared with efficiencies of $8\times 10^{-5}\ M^5$ and $2\times 10^{-4} M^5$ for the mixing noise of subsonic jets with very clean and rather rough exit conditions, respectively (Lighthill 1962). A minimum value of the efficiency (2.19) can perhaps be estimated from the results of Crow & Champagne (1971), which show that, even under carefully controlled upstream conditions, a relative r.m.s. axial fluctuation level of the order of 5×10^{-3} is inevitably developed at the exit plane because of downstream mixing. Of this velocity fluctuation, at least 20 % is contained in a certain preferred instability mode which has S close to unity ($S = \frac{1}{3}\pi$ approximately) and wavelength $4{\cdot}8b$, these estimates being taken from figure 12 and p. 588 of Crow & Champagne (1971). The minimum correlated unsteady thrust can therefore not be less than 10^{-3} times the steady thrust, and the minimum interaction efficiency is then not less than $10^{-6}\ M^3$.

In the high frequency limit $k_0 b \gg 1$, the interaction field cannot be related to any gross features of the exit plane flow, being very much influenced by the rapidly varying pressure fields over the curved walls of the duct. Neither in this limit is it profitable to attempt any quantitative predictions on the basis of (2.11), since that would require unwarranted speculation about the magnitude of the displacement amplitude d at high frequencies. The high frequency directivity can, however, be calculated. The critical result is that when $k_0 b \gg 1$ then $|\alpha| \gg k_0$ and $K_+(-k_0\cos\theta) \propto \sin\frac{1}{2}\theta$, a result which can be seen to follow from the fact that the geometry is now essentially plane, so that $K(s)$ reduces to the appropriate plane form, $K(s) \propto (s^2 - k_0^2)^{\frac{1}{2}}$. Thus $K_+(s) \propto (s+k_0)^{\frac{1}{2}}$ provided that $s = O(k_0)$ while the limits $M \to 0$, $MS = k_0 b \to \infty$ are taken. It then follows from (2.11) that (away from $\theta = \pi$, where the approximations leading to (2.11) are invalid when $k_0 b \gg 1$)

$$I \sim \tan\tfrac{1}{2}\theta, \tag{2.20}$$

showing, as does (2.18) also, a pronounced forward-arc directivity.

3. Modes with azimuthal variation

As has been seen above, it is only in the low frequency limit that a complete parametric description of the sound field, together with a simple interpretation, is possible, Accordingly, we dismiss high frequency oscillations, $k_0 b \gg 1$, with the remark that the directivity pattern of (2.20) continues to hold for them, whatever the order of azimuthal variation.

In the low frequency limit, it can be shown that all modes with azimuthal variation $\exp in(\chi - \overline{\chi})$ say, where $\overline{\chi}$ is a reference angle, generate a scattered field through interaction with the duct lip which is weaker than the field (2.18) by at least a factor M^2S^2, except for the sinuous mode $n = 1$. For that mode the formal analysis of §2 continues to hold, except that the Bessel functions I_0 and

K_0 are to be replaced by I_1 and K_1. In particular, for $n = 1$ and $k_0 b \ll 1$ we find that

$$\phi = -\frac{i\omega bd}{2}\left(\frac{k_0}{\alpha}\right)\left(\frac{b}{R}\exp ik_0 R\right)\left(\frac{K_+(-k_0\cos\theta)}{K_+(\alpha)}\right)\sin\theta,$$

with appropriate modification to (2.8), defining $K(s)$, and with the factor $\exp i(\chi-\overline{\chi})$ understood. Now, however, holding $s = O(k_0)$ and letting $M \to 0$ with S fixed, we have

$$K(s) \sim 1/2b$$

and hence we may take $K_+(-k_0\cos\theta) = K_+(k_0) = K_+(0) = (2b)^{-\frac{1}{2}}$. Then the radiated density field is

$$\rho' = \tfrac{1}{2}\rho_0\left(\frac{b}{R}\exp ik_0 R\right)\left(\frac{k_0}{\alpha}\right)k_0 d\left(\frac{K_+(0)}{K_+(\alpha)}\right)MS\sin\theta. \tag{3.1}$$

The directivity here suggests an interpretation in terms of a dipole at the exit plane with axis transverse to the flow. Now the pressure jump

$$p(x, b-0) - p(x, b+0) = \Delta p$$

is easily calculated in the form

$$\Delta p = \frac{\rho_0 i\omega}{2\pi}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon}\frac{\omega d}{s-\alpha}\frac{K_+(s)}{K_+(\alpha)K(s)}e^{-isx}\,ds$$

and this of course vanishes for $x > 0$. Therefore

$$\begin{aligned}\int_{-\infty}^{0}\Delta p(x)\,dx &= \int_{-\infty}^{+\infty}\Delta p(x)\,dx\\ &= -\frac{i\rho_0\omega^2 d}{\alpha}\frac{K_+(0)}{K_+(\alpha)K(0)}\\ &= -\frac{2i\rho_0\omega^2 bd}{\alpha}\frac{K_+(0)}{K_+(\alpha)}.\end{aligned}$$

Next consider a balance of transverse momentum within a control surface consisting of the exit plane, a distant upstream section through the duct, and the curved duct wall. For $n = 1$ and $k_0 b \ll 1$ all modes inside the duct are exponentially damped, so that there is no contribution from the distant section. The rate of change of transverse momentum within the duct can be estimated, and as in §2 can be shown to be negligible. Therefore the pressure jump integrated along the length of the duct is equivalent to a force per unit length of the rim of the duct, acting radially outwards with magnitude

$$-\frac{2i\rho_0\omega^2 bd}{\alpha}\frac{K_+(0)}{K_+(\alpha)}\exp i(\chi-\overline{\chi}).$$

Resolving this force in an arbitrary direction χ_0 and integrating round the duct rim gives a total force on the duct, concentrated at the rim and with magnitude

$$-\frac{2\pi i\rho_0\omega^2 bd}{\alpha}\frac{K_+(0)}{K_+(\alpha)}\exp i(\chi_0-\overline{\chi}).$$

The force on the fluid is the negative of this, and would give rise to a transverse dipole field which is easily seen to be identical with (3.1).

Thus the field scattered by the sinuous mode can be regarded as the field generated by a dipole at the exit plane, with strength equal to the flux of transverse momentum across the exit plane and with axis normal to the flow direction.

Suppose that the r.m.s. value of the flux of transverse momentum across the exit plane is some fraction ϵ of the steady axial thrust $\rho_0 U^2 A$, then the scattered intensity generated by the sinuous mode is

$$I = \frac{A^2}{16\pi^2 b^2 R^2}\epsilon^2 S^2 \sin^2\theta \rho_0 U^3 M^3, \tag{3.2}$$

and much the same comments apply as in §2.

4. Forward-flight effects

The interpretation of §§2 and 3 in terms of localized multipole sources are especially valuable in that they allow immediate prediction of forward-flight effects. Suppose that the duct moves in the direction of negative x at a flight Mach number M_a, and that, as would be the case relevant in practice, the unsteady levels relative to the duct exit plane are maintained constant. Then we have simply to account for the Doppler amplification in the forward arc due to convective motion of a dipole of prescribed strength at Mach number M_a. (The fact that the monopole strength is itself a time derivative introduces additional Doppler amplification, putting the monopole on precisely the same footing as a dipole from the point of view of convective amplification.) This is achieved (see, e.g. Lighthill 1952) by multiplying each of the intensity fields (2.18) and (3.2) by the factor

$$(1 + M_a \cos\theta)^{-4}. \tag{4.1}$$

In a typical aircraft take-off or landing approach involving values of M_a around 0·3 say, the factor (4.1) represents an increase of around 3 dB and 6 dB at $\theta = 150°$ and $\theta = 180°$ respectively. These values are consistent with the (unpublished) experimental data available to the writer.

In contrast, although jet mixing noise also suffers some small forward-arc amplification, by far the dominant effect is one of an overall reduction due to the reduced shear across the mixing region in forward flight. Forward-flight effects on jet noise are accounted for by a multiplicative factor in the intensity equal to

$$(M - M_a)^7 (1 + M_a \cos\theta)^{-1}$$

(Ffowcs Williams 1963, equation 4.3), in which, under typical conditions, the factor $(M - M_a)^7$ ensures a reduction everywhere in the intensity in forward flight with only a negligible Doppler enhancement due to motion.

5. Conclusions

Instabilities on the shear layer shed from a duct interact with the duct lip to produce an intense sound field. Axisymmetric modes at low frequency drive a fluctuating thrust across the exit plane, and that system constitutes a dipole of axial type at the exit plane. Accompanying the thrust variation is a weak variation in the mass flow (at low Mach number), and this is equivalent to a weak monopole at the exit plane. The strength and phase of the two sources is such that together they produce an intensity distributed in angle as $(1 - \cos\theta)^2$, peaking in the upstream direction and heavily suppressing the field in the rear arc. The intensity varies with the sixth power of exhaust speed, and is increased in the forward arc by forward motion of the duct by the Doppler factor (4.1). An r.m.s. thrust fluctuation of 1 % gives a total scattered power in excess of the power generated by jet mixing noise. The characteristic frequency of the scattered field is set by a preferred instability mode described by Crow & Champagne (1971), and has $\omega b/U$ close to unity. In real engines, this frequency exceeds that at which mixing noise peaks by a factor of around 4, depending very much upon the definitions used.

Fluctuating thrust has been proposed before (Ffowcs Williams & Gordon 1965; Ffowcs Williams 1968) as a possibly dominant source of low speed jet noise. However, the arguments used in support of this claim have not taken into account the presence of mean-flow and of shear-layer instabilities, neither have they recognized the presence of the weak monopole which is coupled to the fluctuating thrust. Without that recognition, the dipole alone would give a peak field in the downstream direction, which would be hopelessly confused with the rear-arc mixing noise. It is hoped that the calculation given here will go some way towards a proper modelling of the acoustic effects of unsteady exit flow conditions.

Interaction of the first-order sinuous mode of instability with the nozzle also produces a scattered field with a dependence upon the sixth power of velocity. This field peaks in the sideline direction, $\theta = 90°$, and is there unaffected by forward flight. The dominance in many engines of a field at 90° which depends only on the exhaust speed relative to the nozzle suggests an explanation in terms of that mechanism, allied with visual evidence of the presence of the sinuous mode of instability. Again a simple interpretation can be given; the sinuous mode drives a fluctuating transverse thrust (though no axial thrust or mass flow) across the exit plane, which constitutes a transverse dipole. A correlated unsteadiness of the order of 1 % is quite sufficient to overwhelm the mixing noise in the sideline direction.

We should emphasize that the parametric variations $I \sim U^6\,(1-\cos\theta)^2$ and $I \sim U^6 \sin^2\theta$ of (2.18) and (3.3) were first derived by Leppington (Chapter 4 of Ffowcs Williams *et al.* 1972) in the solution of the problem of interaction of a Lighthill-type quadrupole with a semi-infinite circular duct, with neglect of the jet flow and its associated instabilities. Leppington did not, however, interpret these laws in the multipole fashion which seems essential for quantitative estimates of the scattered field and of forward-flight effects, to be given. Another point which should be emphasized is that the solutions given here do not enforce a Kutta condition on the unsteady flow at the rim of the duct. The pressure jump across the duct wall does of course vanish as the rim is approached (and quasi-steady theories of flow through cascades of aerofoils would regard that as a sufficient Kutta condition) but nonetheless here, as in most diffraction problems, the velocity components have mild singularities at the rim, and the gradient of the vortex layer becomes mildly infinite. Whether or not the frequencies of interest in aerodynamic noise are low enough to justify the imposition of a stronger condition (that the velocities should vanish at the rim and that the shear layer should leave the rim with zero gradient) is a completely open question. The author does not believe that a stronger condition should be imposed, and has in any case been unable to find a solution satisfying the full Kutta condition for this problem. We remark, however, that the enforcement of the full Kutta condition in the case of a vortex sheet leaving a flat plate results in great changes in the sound field (Crighton 1972), the intensity law being $I \sim U^4 \sin \frac{1}{2}\theta$ with no Kutta condition and $I \sim U^2 \operatorname{cosec}^2 \frac{1}{2}\theta$ under the full Kutta condition.

It will be seen from this discussion that we have found a noise source producing a field with all the properties (*a*)–(*e*) of § 1 which we have here called *the* 'excess noise' field. By this, however, we intend no implication that these results are sufficient to explain all 'excess noise', meaning by that all deviations, from the predictions of the Lighthill theory of pure jet mixing noise, of the noise fields of all real engines or rigs. These deviations are so obviously different in different cases that there can be no single theory of excess noise. On the other hand, it seems that there is a certain degree of universality in excess noise, represented by a field with attributes (*a*)–(*e*). If that is the case, the only cure seems to lie in the reduction of correlated unsteadiness across the exit plane, either by careful design or struts and other devices in the engine tailpipe, by the control of shear-layer instabilities which will *drive* exit plane unsteadiness regardless of upstream conditions, or by the introduction of devices in the tailpipe which reduce turbulence levels and correlation scales. It is perhaps interesting here to look at the proposal for control of jet mixing noise which is implicit in the work of Crow & Champagne (1971). The idea emerges from their work that mixing in the first eight diameters of the jet can be turned into an orderly process if a slight coherent periodic forcing at an appropriate Strouhal number is generated at the exit plane. Any attempt to control mixing noise in this way would, if the work described here is correct, produce a disastrous increase in the forward-arc excess noise. Viewed in this light, an examination of the acoustic field induced by the big-eddy control mechanism of Crow & Champagne seems badly needed.

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